The Gribov-Zwanziger action and its Schwinger-Dyson equations

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The Gribov region

- Yang-Mills theory in Landau gauge

$$S_{FP} = \int d^D x \, \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \left(b^a \partial_\mu A^a_\mu + \bar{c}^a \partial_\mu D^{ab}_\mu c^b \right)$$

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$$\partial_\mu A_\mu = 0$$

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- Gribov copies still remain in Ω

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- Working hypothesis

$$Z_\Lambda \sim Z_\Omega$$

The horizon action

- Implement the Gribov region for SU(N) in D dimensions through the means of the horizon function $h(x)$

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- Non-local horizon term, introduction of 2 pairs of new complex fields in quantization :

$(\bar{\varphi}^{ac}_\mu, \varphi^{ac}_\mu)$ a pair of complex conjugate bosonic fields

$(\bar{\omega}^{ac}_\mu, \omega^{ac}_\mu)$ a pair of Grassmann fields

The Gribov-Zwanziger action

- Horizon term written as a Gaussian integration over the new fields

$$
\begin{aligned}
S_{GZ} &= S_{YM} + S_{gf} \\
&+ \int d^D x \left[\partial_\lambda \bar{\varphi}^{ac}_\mu (D_\lambda \varphi_\mu)^{ab} - \partial_\lambda \bar{\omega}^{ac}_\mu (D_\lambda \omega_\mu)^{ab} \right] \\
&+ \int d^D x \left[\gamma^{\frac{1}{2}} (D_\lambda (\varphi_\lambda - \bar{\varphi}_\lambda))^{aa} \right]
\end{aligned}
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- φ and ω are new ghost fields with opposed statistics. Let $\varphi = \frac{1}{\sqrt{2}}(U + iV)$, where V_μ^{ac} and U_μ^{ac} are real fields

The Gribov-Zwanziger action - continued

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$$S_{GZ} = S_{YM} + S_{gf} + \int d^D x \left[\frac{1}{2} \left(\partial_\lambda V_\mu^{ac} (D_\lambda V_\mu)^{ab} + \partial_\lambda U_\mu^{ac} (D_\lambda U_\mu)^{ab} \right) \right] - \int d^D x \left[\partial_\lambda \bar{\omega}_\mu^{ac} (D_\lambda \omega_\mu)^{ab} \right] + \int d^D x \left[i\sqrt{2}\gamma^{\frac{1}{2}} (D_\lambda V_\lambda)^{aa} \right]$$

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$$- \int d^D x \left[\partial_\lambda \bar{\omega}_\mu^{ac} (D_\lambda \omega_\mu)^{ab} \right] + \int d^D x \left[i\sqrt{2} \gamma^{\frac{1}{2}} (D_\lambda V_\lambda)^{aa} \right]$$

- γ is given by the gap equation

$$\langle 2igf^{abc} A_\mu^a V_\mu^{bc} \rangle = 2\gamma^2 D(N^2 - 1)$$

where the mean is taken with the Z_{GZ} partition function

The different fields of S_{GZ}

Field	Degrees of freedom	Statistic
$c, \bar{c}$	1	fermionic
$\omega, \bar{\omega}$	$D(N^2-1)$	fermionic
U	$D/2(N^2-1)$	bosonic
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- $(\bar{\omega}^{ac}_{\mu}, \omega^{ac}_{\mu})$, $(\bar{c}^{a}_{\mu}, c^{a}_{\mu})$ and U^{ac}_{μ} act as regular Faddeev-Popov ghosts in closed loops
- V^{ac}_{μ} couples directly to the gluon field A through the γ term

Propagators and two-point functions

- The Dyson-Schwinger equations allow us to compute the various two-point correlation functions

$$\frac{\delta^2\Gamma_{GZ}}{\delta A^a_\mu(x)\delta A^b_\nu(y)} \equiv \Gamma_{AA} \qquad \frac{\delta^2\Gamma_{GZ}}{\delta V^{ab}_\mu(x)\delta V^{cd}_\nu(y)} \equiv \Gamma_{VV}$$

$$\frac{\delta^2\Gamma_{GZ}}{\delta V^{ab}_\mu(x)\delta A^c_\nu(y)} \equiv \Gamma_{VA}$$

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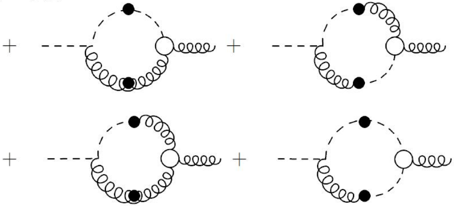
- When cast into a "two-point function" matrix, we have a simple matrix relation with the propagator matrix

$$\begin{pmatrix} \Gamma_{AA} & \Gamma_{AV} \\ \Gamma_{AV} & \Gamma_{VV} \end{pmatrix}^{-1} = \begin{pmatrix} D_{AA} & D_{AV} \\ D_{AV} & D_{VV} \end{pmatrix}$$

The Γ_{AV} matrix element

- The Γ_{AV} function's diagrammatic expression is given by

$$\frac{\delta^2 \Gamma_{GZ}}{\delta V^{ab}_{\mu}(x)\delta A^{c}_{\nu}(y)} = i\sqrt{2}\gamma^{\frac{1}{2}} g f^{abc} \delta_{\mu\nu}\delta(x-y)$$



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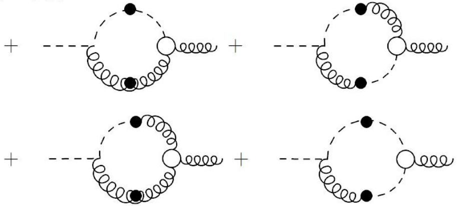

- Includes fully dressed vertices and propagators (!)

The tree-level and color approximations

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- Take out the color structure by projecting on the adjoint space of SU(3)
- Project onto the transverse part of the external momentum k

$$\Gamma_{AV}(k)\mathcal{P}_{\mu\nu}(k)\delta^{nc} = ig\sqrt{2}\gamma^{\frac{1}{2}}\delta_{\mu\nu} + \frac{g^2}{\sqrt{\Lambda}} f^{aed} f^{kci} f^{bna} \int \frac{d^D q}{(2\pi)^D}\, q_\nu D^{eil}_{A_\lambda V_\rho}(q) D^{kldb}_{V_\rho V_\mu}(k+q) k_\lambda$$

written in terms of the D_{AV} and D_{VV} propagators

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IR propagator's behavior

$$D_{AA}(k) \approx \frac{c_{AA}}{(k^2)^{1+\alpha_{AA}}} \quad D_{VV}(k) \approx \frac{c_{VV}}{(k^2)^{1+\alpha_{VV}}}$$

$$D_{VA}(k) \approx \frac{c_{AV}}{(k^2)^{1+\alpha_{AV}}}$$

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- c_{AA}, c_{VV} and c_{AV} are dimensionless parameters
  α_{AA}, α_{VV} and α_{AV} are the infrared exponents (IREs)

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$$\Gamma_{AV}(k) \propto -c_{AV}\; c_{VV}\; (k^2)^{-1-\alpha_{AV}-\alpha_{VV}+D/2}\, \mathcal{I}_3(\alpha_{AV}, \alpha_{VV})$$

where $\mathcal{I}_3(\alpha_{AV}, \alpha_{VV})$ is a loop integral depending on two IREs

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where $\mathcal{I}_3(\alpha_{AV}, \alpha_{VV})$ is a loop integral depending on two IREs

$$\mathcal{I}_3(\alpha_{AV}, \alpha_{VV}) = \frac{1}{(4\pi)^{\frac{D}{2}}} \left(\frac{D-1}{2}\right) \left[\frac{D-1}{2} - \alpha_{VV}\right] \times$$
$$\frac{\Gamma\left(2+\alpha_{AV}+\alpha_{VV}-D/2\right)}{\Gamma(2+\alpha_{VV})\Gamma(2+\alpha_{AV})} \beta\left(-\alpha_{AV}+D/2, -\alpha_{VV}+D/2\right)$$

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$$\Gamma_{AA} \propto (k^2)^{-1-2\alpha_{AV}+D/2} \left(c_{AA}c_{VV} \, \mathcal{I}_1 + c_{AV}^2 \, \mathcal{I}_2\right)$$

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- We assumed here that fields also obey power laws, which gives a very simple first relation between the IREs

A relation between IREs

$$2\alpha_{AV} = \alpha_{AA} + \alpha_{VV}$$

Inversing the Γ matrix

- Inversing the matrix relation between two-point functions and propagators. . .

$$\begin{pmatrix} \Gamma_{AA} & \Gamma_{AV} \\ \Gamma_{AV} & \Gamma_{VV} \end{pmatrix} = \frac{1}{D_{AA}D_{VV} - D_{AV}^2} \begin{pmatrix} D_{VV} & -D_{AV} \\ -D_{AV} & D_{AA} \end{pmatrix}$$

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Another relation between IREs

$$\frac{1}{2}\,\mathcal{I}_4(\alpha_{VV}) = \mathcal{I}_3(\alpha_{AV}, \alpha_{VV})$$

The $\mathcal{D}$ matrix determinant

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A third relation between IREs

$$\mathcal{I}_1(\alpha_{AA}, \alpha_{VV}) + \mathcal{I}_2(\alpha_{AV}) = \mathcal{I}_3(\alpha_{AV}, \alpha_{VV})$$

Solving for the IREs

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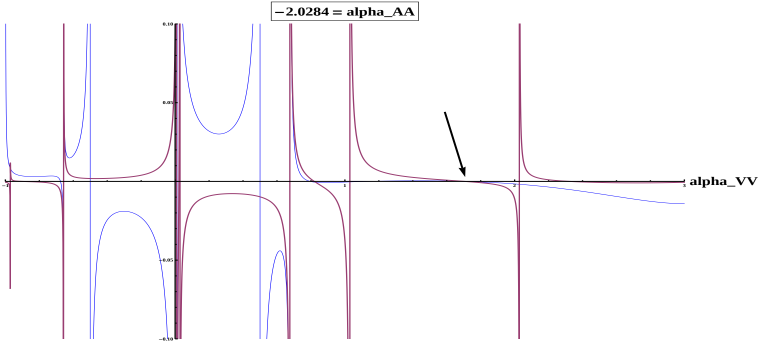

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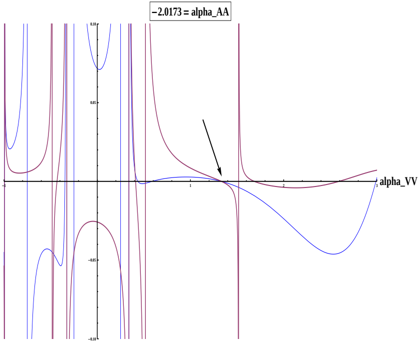

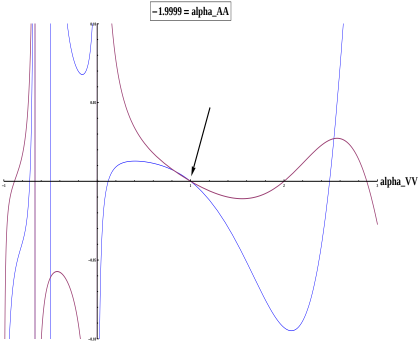

The IREs values

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IREs

$D = 4$	$D = 3$	$D = 2$
$\alpha_{AA} \approx -2.028$	$\alpha_{AA} \approx -2.017$	$\alpha_{AA} \approx -2$
$\alpha_{VV} \approx 1.706$	$\alpha_{VV} \approx 1.332$	$\alpha_{VV} \approx 1$
$\alpha_{AV} \approx -0.161$	$\alpha_{AV} \approx -0.343$	$\alpha_{AV} \approx 1/2$

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- Still a lot of approximations were made !