On QCD and effective locality

T. Grandou[‡]

collaborations with

M. Gattobigio[‡], Y.-M. Sheu[†‡], & H. M. Fried[†]

[†]Brown University

[‡]Institut Non Linéaire de Nice Sophia-Antipolis, UMR-CNRS 6618

Abstract

In a recent paper it was shown how quark scattering in a quenched, eikonal model led to a momentum-transfer dependent amplitude expressed in terms of Halpern's functional integral; and how the requirement of manifest gauge invariance converted that functional integral into a local integral, capable of being evaluated with precision by a finite set of numerical integrations.

We here prove that this property of "effective locality" holds true for all quark processes, without approximation and without exception.

Linkage operator

- QED Generating functional: using functional differentiation instead of functional integral

$$Z_{\rm QED}[j,\eta,\bar\eta] = \langle \mathbf{S}\rangle^{-1}\, e^{\frac{i}{2}\int j\cdot \mathbf{D}_c\cdot j}\,\cdot \boxed{e^{\mathfrak{D}_A}}\cdot\, e^{i\int \bar\eta\cdot\mathbf{G}_c[A]\cdot\eta+\mathbf{L}[A]}\Big|_{A=\int \mathbf{D}_c\cdot j}$$

- Green's function(al) : $\mathbf{G}_c[A] = [m+\gamma\cdot(\partial - igA)]^{-1} = \mathbf{S}_c\cdot[1-g\,(\gamma\cdot A)\,\mathbf{S}_c]^{-1}$
- Closed-Fermion-loop functional : $\mathbf{L}[A] = \mathbf{Tr}\,\ln\,[1-ig(\gamma\cdot A)\mathbf{S}_c]$
- The linkage operator $e^{\mathfrak{D}_A}$ with $\mathfrak{D}_A = -\frac{i}{2}\int \frac{\delta}{\delta A}\cdot\mathbf{D}_c\cdot\frac{\delta}{\delta A}$

$$\mathbf{S}'_c(x,y) = e^{\mathfrak{D}_A}\cdot\left[\mathbf{G}_c(x,y|A)\,\frac{e^{\mathbf{L}[A]}}{\langle\mathbf{S}\rangle}\right]\Bigg|_{A=0}$$

$$\mathbf{M}_c(x_1,x_2;y_1,y_2) = i^2\, e^{\mathfrak{D}_A}\cdot\left[\mathbf{G}^{\rm I}_c(x_1,y_1|A)\,\mathbf{G}^{\rm II}_c(x_2,y_2|A)\,\frac{e^{\mathbf{L}[A]}}{\langle\mathbf{S}\rangle}\right]\Bigg|_{A=0}$$

- The linkage operator connects two A_μ-fields, replacing them by a photon propagator. At all orders of g, it inserts photon propagators between fermion propagators and closed-fermion loops.

QCD Generating Functional

$$\mathcal{L}^{(0)}_{\rm QCD} = -\bar{\psi}\,[m + \gamma_\mu\,\partial_\mu]\,\psi - \frac{1}{4}\mathbf{f}^{\mu\nu}_a\mathbf{f}^a_{\mu\nu} - \frac{\mathbf{1}}{\mathbf{2}\zeta}(\partial_\mu\mathbf{A}^{\mathbf{a}}_\mu)^{\mathbf{2}}$$

$$\mathcal{L}'_{\rm QCD} = +ig\,\bar{\psi}\,(\gamma_\mu\,A^a_\mu\lambda^a)\,\psi - \frac{1}{4}(F^{\mu\nu}_a F^a_{\mu\nu} - \mathbf{f}^a_{\mu\nu}\mathbf{f}^a_{\mu\nu}) + \frac{\mathbf{1}}{\mathbf{2}\zeta}(\partial_\mu\mathbf{A}^{\mathbf{a}}_\mu)^{\mathbf{2}}$$

• Adding & subtracting a gauge fixing term : the overall QCD action manifestly gauge invariant and Lorentz covariant:

QCD Generating Functional

- Generating functional

$$\mathcal{Z}_{QCD}[j,\eta,\bar{\eta}] \quad = \quad e^{\frac{i}{2}\int j\cdot\mathbf{D}_c^{(\zeta)}\cdot j}\mathcal{N}\int \mathrm{d}[\chi]\, e^{\frac{i}{4}\int\chi^2}\, e^{\mathfrak{D}_A}\, e^{\frac{i}{2}\int\chi\mathbf{F}+\frac{i}{2}\int A\cdot\left(\mathbf{D}_c^{(\zeta)}\right)^{-1}\cdot A} \tag{1}$$

$$\cdot\, e^{i\int\bar{\eta}\cdot\mathbf{G}_c[A]\cdot\eta}\, e^{\mathbf{L}[A]}\Big|_{A=\int\mathbf{D}_c^{(\zeta)}j},$$

where

$$\mathfrak{D}_A = -\frac{i}{2}\int\frac{\delta}{\delta A_\mu^a}\,\mathbf{D}_c^{(\zeta)}\Big|_{\mu\nu}^{ab}\frac{\delta}{\delta A_\nu^b} \tag{2}$$

and

$$\mathbf{F}_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc}A_\mu^b A_\nu^c. \tag{3}$$

and

$$\mathbf{D}_c^{(\zeta)}\Big|_{\mu\nu}^{ab} = \delta^{ab}(-\partial^2)^{-1}\left[g_{\mu\nu} - \zeta\frac{\partial_\mu\partial_\nu}{\partial^2}\right] \tag{4}$$

Halpern's field

We have used

$$e^{-\frac{i}{4}\int F^a_{\mu\nu}F_a^{\mu\nu}} = \mathcal{N}\int d[\chi]\, e^{\frac{i}{4}\int \chi^a_{\mu\nu}\chi_a^{\mu\nu} + \frac{i}{2}\int \chi^a_{\mu\nu}F_a^{\mu\nu}} \tag{5}$$

and

$$\int \mathrm{d}[\chi] = \prod_i \prod_a \prod_{\mu>\nu} \int d\chi^a_{\mu\nu}(w_i) \tag{6}$$

Schwinger-Fradkin Representation

- Green's function(al): $\mathbf{G}_c[A] = \left[m + \gamma_\mu \left(\partial_\mu - igA^a_\mu \, \lambda^a\right)\right]^{-1}$
- Fermion closed-loop functional: $\mathbf{L}[A] = \mathbf{Tr} \ln \left[1 - ig\gamma \cdot A \cdot \lambda \mathbf{S}_c\right]$

- Example of a full and exact Fradkin representation of $\mathbf{G}_c[A]$:

$$\langle p|\mathbf{G}_c[A]|y\rangle = e^{-ip\cdot y} \cdot i \int_0^\infty ds \, e^{-ism^2} \cdot e^{-\frac{1}{2}\mathbf{Tr}\ln(2h)}$$
$$\times \int d[u]\{m - i\gamma \cdot [p - gA(y - u(s))]\} \cdot e^{\frac{i}{4}\int_0^s ds' \, [u'(s')]^2} \cdot e^{ip\cdot u(s)}$$
$$\times \boxed{\left(e^{g\int_0^s ds' \sigma\cdot F(y-u(s'))} \cdot e^{-ig\int_0^s ds' \, u'(s')\cdot A(y-u(s'))}\right)_+}$$

- Its Bloch-Nordsieck approximation :

$$\langle p|\mathbf{G}_c^{\mathrm{BN}}[A]|y\rangle = e^{-ip\cdot y} \cdot i \int_0^\infty ds \, e^{-is(m^2+p^2)} \, \left\{m - i\gamma_\mu \, \left[p_\mu - g\, A^a_\mu(y + 2sp)\lambda^a\right]\right\}$$
$$\times \boxed{\left(\exp\left\{g\int_0^s ds'\sigma^{\mu\nu} \, F^a_{\mu\nu}(y + 2s'p)\lambda^a\right\} \cdot \exp\left\{+2ig\int_0^s ds' \, \left[p^\mu \, A^a_\mu(y + 2s'p)\lambda^a\right]\right\}\right)_+}$$

All $Q/\bar{Q}$ amplitudes ..

- $p_1 + p_2 \to p_1' + p_2'$:

are obtained by pair-wise functional differentiation of Z_{QCD} w.r.t quark sources η^a_μ, $\bar{\eta}^b_\nu$. Each such operation "brings down" one (of a set of possible properly anti-symmetrized) Green's functions $\mathbf{G}_c[A]$. For example :

$$\mathcal{N} \int \mathrm{d}[\chi]\, e^{\frac{i}{4}\int \chi^2}\, e^{\mathfrak{D}\, A}\, e^{\frac{i}{2}\int \chi \mathbf{F} + \frac{i}{2}\int A \left(\mathbf{D}_c^{(\zeta)}\right)^{-1} A} \tag{7}$$

$$\times \left[\mathbf{G}_c^{\mathrm{I}}(x_1, y_1|A)\, \mathbf{G}_c^{\mathrm{II}}(x_2, y_2|A)\, \frac{e^{\mathbf{L}[A]}}{\langle \mathbf{S} \rangle} \right]\Bigg|_{A=0}$$

Gathering gluon field dependences ..

The linear A_μ-field dependences can be taken out of the ordered exponentials defining the $G_c^{I,II}[A]$, the pieces

$$\boxed{\left(e^{g\int_0^s ds' \sigma\cdot F(y-u(s'))} \cdot e^{-ig\int_0^s ds'\, u'(s')\cdot A(y-u(s'))}\right)_+} \tag{8}$$

and likewise for the quadratic spinorial- contributions, the pieces

$$\left(\exp\left[g\int_0^s ds'\,\sigma_{\mu\nu}\lambda^a f^{abc} A_\mu^b(z) A_\nu^c(z)\right]\right)_+, \quad z = y - u(s'). \tag{9}$$

For example :

$$\left(e^{2ig\int_{-\infty}^{+\infty} ds\, p_\mu A_\mu^a(y+2sp)\lambda^a}\right)_+$$
$$= \int \mathfrak{D}\alpha\,\delta\left[\alpha^a(s) - 2gp_\mu A_\mu^a(y+2sp)\right]\left(e^{i\int_{-\infty}^{+\infty} ds\,\alpha^a(s)\lambda^a}\right)_+$$

Letting $e^{\mathfrak{D}A}$ operate ..

One gets

$$e^{-\frac{i}{2}\int \frac{\delta}{\delta A}\cdot \mathbf{D}_{\mathrm{c}}^{(\zeta)}\cdot \frac{\delta}{\delta A}} \cdot e^{+\frac{i}{2}\int A\cdot\bar{\mathcal{K}}\cdot A + i\int \bar{\mathcal{Q}}\cdot A} \cdot e^{\mathbf{L}[A]} \tag{10}$$

with

$$\bar{\mathcal{K}}^{ab}_{\mu\nu} = \mathcal{K}^{\mathbf{ab}}_{\mu\nu} + [\, g f^{abc}\chi^{c}_{\mu\nu} + \left(\mathbf{D}_{\mathrm{c}}^{(\zeta)\,-1}\right)^{ab}_{\mu\nu}] \tag{11}$$

and

$$\bar{\mathcal{Q}}^{a}_{\mu} = -\partial_\nu \chi^{a}_{\mu\nu} + \mathcal{Q}^{a}_{\mu} = -\partial_\nu \chi^{a}_{\mu\nu} + g[\mathcal{R}^{a}_{I\,\mu} + \mathcal{R}^{a}_{II\,\mu}] \tag{12}$$

A functional identity is required :

$$e^{\mathfrak{D}\,A} \cdot (\mathcal{F}_{\mathrm{I}}[A]\,\mathcal{F}_{\mathrm{II}}[A]) = \left(e^{\mathfrak{D}\,A}\cdot \mathcal{F}_{\mathrm{I}}[A]\right)\cdot e^{\overleftrightarrow{\mathfrak{D}}}\cdot \left(e^{\mathfrak{D}\,A'}\cdot \mathcal{F}_{\mathrm{II}}[A']\right)\Big|_{A'=A} \tag{13}$$

with the "cross-linkage" operator :

$$\overleftrightarrow{\mathfrak{D}} = -i\int \overleftarrow{\frac{\delta}{\delta A}}\cdot \mathbf{D}_{\mathrm{c}}^{(\zeta)}\cdot \overrightarrow{\frac{\delta}{\delta A'}}, \tag{14}$$

With ..

$$\mathcal{F}_{\mathrm{I}}[A] = \exp\left[\frac{i}{2}\int A\cdot\bar{\mathcal{K}}\cdot A + i\int\bar{\mathcal{Q}}\cdot A\right], \quad \text{and} \qquad \mathcal{F}_{\mathrm{II}}[A] = \exp\left(\mathbf{L}[A]\right) \tag{15}$$

one gets, for $\left(e^{\mathfrak{D}\,A}\cdot\mathcal{F}_{\mathrm{I}}[A]\right)$, the (improved) previous result of

$$\exp\left[\frac{i}{2}\int\bar{\mathcal{Q}}\cdot\mathbf{D}_c^{(\zeta)}\cdot\left(1-\bar{\mathcal{K}}\cdot\mathbf{D}_c^{(\zeta)}\right)^{-1}\cdot\bar{\mathcal{Q}} + \frac{1}{2}\mathbf{Tr}\ln\left(1-\bar{\mathcal{K}}\cdot\mathbf{D}_c^{(\zeta)}\right)\right] \tag{16}$$
$$\cdot\exp\left[\frac{i}{2}\int A\cdot\bar{\mathcal{K}}\cdot\left(1-\mathbf{D}_c^{(\zeta)}\cdot\bar{\mathcal{K}}\right)^{-1}\cdot A + \int\bar{\mathcal{Q}}\cdot\left(1-\mathbf{D}_c^{(\zeta)}\cdot\bar{\mathcal{K}}\right)^{-1}\cdot A\right],$$

where

$$1-\hat{\mathcal{K}}\cdot\mathbf{D}_c^{(\zeta)} - {\mathbf{D}_c^{(\zeta)}}^{-1}\cdot\mathbf{D}_c^{(\zeta)} = -\hat{\mathcal{K}}\cdot\mathbf{D}_c^{(\zeta)} \tag{17}$$

and, now, $\hat{\mathcal{K}} = \mathcal{K} + g(f\cdot\chi)$

Then, the string of operators ..

$$\exp\left[-\frac{i}{2}\int \bar{\mathcal{Q}}\cdot\hat{\mathcal{K}}^{-1}\cdot\bar{\mathcal{Q}}+\frac{1}{2}\mathbf{Tr}\ln\left(\hat{\mathcal{K}}\right)+\frac{1}{2}\mathbf{Tr}\ln\left(-\mathbf{D}_c^{(\zeta)}\right)\right] \tag{18}$$

$$\cdot\exp\left[\frac{i}{2}\int\frac{\delta}{\delta A'}\cdot\mathbf{D}_c^{(\zeta)}\cdot\frac{\delta}{\delta A'}\right]\cdot\exp\left[\frac{i}{2}\int\frac{\delta}{\delta A'}\cdot\hat{\mathcal{K}}^{-1}\cdot\frac{\delta}{\delta A'}-\int\hat{\mathcal{Q}}\cdot\bar{\mathcal{K}}^{-1}\cdot\frac{\delta}{\delta A'}\right]$$

is to operate on

$$\left(e^{\mathfrak{D}_{A'}}\cdot\mathcal{F}_{\mathbb{I}}[A']\right)$$

giving essentially

$$\exp\left[-\frac{i}{2}\int \bar{\mathcal{Q}}\cdot\hat{\mathcal{K}}^{-1}\cdot\bar{\mathcal{Q}}+\frac{1}{2}\mathbf{Tr}\ln\left(\hat{\mathcal{K}}\right)\right] \tag{19}$$

times the operation

$$e^{\frac{i}{2}\int\frac{\delta}{\delta A}\cdot\hat{\mathcal{K}}^{-1}\cdot\frac{\delta}{\delta A}}\cdot e^{-\int\bar{\mathcal{Q}}\cdot\hat{\mathcal{K}}^{-1}\cdot\frac{\delta}{\delta A}}\cdot e^{\mathbf{L}[A]} \tag{20}$$

TO ALL ORDERS IN g, AND EVERY SIMILAR PROCESS

.. and nothing in this result ..

$$\exp\left[-\frac{i}{2}\int \bar{\mathcal{Q}}\cdot\hat{\mathcal{K}}^{-1}\cdot\bar{\mathcal{Q}}+\frac{1}{2}\mathbf{Tr}\ln\left(\hat{\mathcal{K}}\right)\right] \tag{21}$$

$$\times\ e^{\frac{i}{2}\int\frac{\delta}{\delta A}\cdot\hat{\mathcal{K}}^{-1}\cdot\frac{\delta}{\delta A}}\cdot e^{-\int\bar{\mathcal{Q}}\cdot\hat{\mathcal{K}}^{-1}\cdot\frac{\delta}{\delta A}}\cdot e^{\mathbf{L}[A]} \tag{22}$$

ever refers to $\mathbf{D}_c^{(\zeta)}$! Remember : $\hat{\mathcal{K}}=\mathcal{K}+g(f\cdot\chi)$. That is :

GAUGE-INVARIANCE IS HERE ACHIEVED AS A MATTER OF GAUGE-INDEPENDENCE

Moreover this result is effectively local, thanks to

$$\langle x|\hat{\mathcal{K}}^{-1}|y\rangle=\hat{\mathcal{K}}^{-1}(x)\,\delta^{(4)}(x-y)$$

and likewise for the linear $\mathcal{Q}$ and quadratic pieces $\mathcal{K}$ of $\mathbf{L}[A]$. So that terms in (21) and (22) depend only on simple combinations of y and Fradkin's variables $u(s')$ in a specific (Ex : $y-u(s')$) still local way .

A result at variance with the non-manifestly gauge-invariant QED ..

.. where the carriers of each interaction are the " action at distance" $\mathbf{D}_c(w-z)$

w and $z = y - u(s')$: relevant space-time and Fradkin variables.

In QCD instead, and at least for all of those $Q/\bar{Q}$ amplitudes,

$\mathbf{D}_c(w-z)$ gets replaced by the local, "contact-type" interaction term $\hat{\mathcal{K}}^{-1}(w) \times \delta^{(4)}(w-z)$.

In some cases ..

.. interesting simplifications show up (Example : Eikonal +BN)

$$
\begin{aligned}
& \delta^{(4)}(w - y_1 + s_1 p_1) \cdot \delta^{(4)}(w - y_2 + s_2 p_2) \\
= \quad & \frac{1}{2pE} \cdot \delta^{(2)}(\vec{y}_{1,\perp} - \vec{y}_{2,\perp}) \cdot \delta(s_1 - s_+) \cdot \delta(s_2 - s_-) \\
& \cdot \delta^{(2)}(\vec{w}_\perp - \vec{y}_\perp) \cdot \delta\left(w_L - \frac{1}{2}\left(y_{1,L} + y_{2,L}\right)\right) \cdot \delta\left(w_0 - y_{1,0} + \frac{E}{p}\, y_{1,L}\right)
\end{aligned}
$$

In CM,

$$
\begin{aligned}
\vec{y}_\perp = \vec{y}_{1,\perp} = -\vec{y}_{2,\perp} \equiv \frac{1}{2}\vec{b} \\
z_0 = y_{1,0} - y_{2,0} = 0 \quad \longrightarrow \quad s_1 = s_2 \\
y_{1,0} = \gamma\, m\, s_1 \quad \longrightarrow \quad s_1 = y_{1,0}/(\gamma\, m)
\end{aligned}
$$

For large γ and any reasonable duration of the scattering, $s_1 = s_2 \approx 0$

$$
ig\delta^{(2)}(\vec{b})\, \Omega^a_{\mathrm{I}}(0)\ \left[f \cdot \chi(w^{(0)}) \right]^{-1} \Big|^{ab}_{30}\ \Omega^b_{\mathrm{II}}(0),
$$

where $w^{(0)}_\mu = (\vec{y}_\perp, \vec{0}_L; y_0)$ for $E/p \approx 1$.

A proviso ..

.. all this refers to particles . The longitudinal momenta of quarks can be estimated, not the transverse ones.

Treated by replacing $\delta^{(2)}(\vec{y}_{1\perp} - \vec{y}_{2\perp}) = \delta^{(2)}(\vec{b})$ by

$$(2\pi)^2 \int \mathrm{d}^2\vec{k}_\perp \, e^{i\vec{k}_\perp\cdot\vec{b} - \vec{k}_\perp^2/M^2} = \frac{M^2}{4\pi} \exp\left[-\frac{M^2\vec{b}^2}{4}\right]$$

where $M \simeq \mathcal{O}$(total CM scattering energy).

$$\begin{aligned} s_1 &= s_2 = s \simeq 0 \\ w_\mu &= w_\mu^{(0)} \\ \chi &= \chi(w_\mu^{(0)}) \end{aligned}$$

Prospects

- Removal of ordered exponentials at high energy and Eikonal approximation $\rightarrow$ numerical calculations.
- Analyses of renormalization effects (color charge, ..).
- Extension to other (gluonic) processes
- Analyses of strong and weak coupling limits

much work and explorations to do : collaborators welcome !