# ANPAEQED II. : A Mechanism for Finite Charge Renormalization

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1) Definition of ANPAEQED

2) Contents of I : "The two-point Functions" (Phys. Rev. D79 (2009) 065035)
"Quenched" approximations for $S_c'$, $D'_{c\mu\nu}$ :

$S_c' \to \sum$  , $D_c' \to \sum$  .

Secret(s) of I and II :

a) Schwinger/Symanzik functional solú (in modified format)
b) Fradkin's Representations (in modified format) of :

$G_c[A]$ = Pot. Theory $G_c$ for electron propagation in $A_\mu(x)$,

$$G_c[A] = S_c\,[1 - ig_0\gamma\cdot A\, S_c]^{-1}\ , \quad S_c = \text{free elec. prop.};$$

and

$$L[A] = \log \text{ of "fermion determinant"} = \mathrm{Tr}\,\ln[1 - ig_0\gamma\cdot A\, S_c]\ ,$$

or: $L[A] = \sum_{2n}$  .

3) Object of Non-Pert Studies: Sums over all relevant Feyn. graphs.

Traditionally, attempted by $\Sigma$ over a "subclass" of FGs :

viz.: $\sum$  , or: 

· Basically incorrect, because $\sum (g^2)^n$ introduce a new class of FGs with each increase of $n$ ; and each class contains an $\infty$ number of FGs, as $n$ increases.

## Previous Attempts at Finite $Z_3^{-1}$ :

Notation : If $\alpha = \frac{e^2}{4\pi}$ , $\alpha_0 = \frac{e_0^2}{4\pi}$ , renormalization theory shows that : $\alpha = \alpha_0 Z_3$ .

Since $Z_3(\alpha_0)$, if $Z_3^{-1} < \infty$, then (probably) $\alpha_0 < \infty$; and then, $\alpha$ is calculable.
Efforts made to sum sub-classes/approximations to Feynman graphs; to show that the UV log divergences cancel; and then obtain : $\alpha = 1/137$.

#1 : Johnson, Baker, Wiley : Required special assumptions (only single log divs; $m_0 = 0$; pert. approxi. to B-S kernals). Result : ↓ (late '60s)

#2 : In 2004 : Gies and Jaeckel considered a num. analysis of "Renorm. Flow of QED", making approxs. to photon-field fluctuations. Result : ↓

Our Method : Identify and extract log. divs. at the functional level, with the aid of manifest gauge-invariance built into Fradkin representation for $L[A]$; and $\Sigma$ all divs. before evaluating Functional integrals (FIs).

Tremendous bonus : Pert. cancellations in every pert. order can be seen within context of Fradkin rep, and cancellations performed before evaluation of FIs.

## Schwinger/Symanzik Functional Solu's for $D_c'^{\mu\nu}$ :

(modified but exact)

$$D'_{c,\mu\nu} = D_{c,\mu\nu} + \iint D_{c,\mu\lambda} K_{\lambda\sigma} D_{c,\sigma\nu} ,$$

$$K_{\mu\nu}(x-y) = -i \frac{\delta}{\delta A_\mu(x)} \cdot \frac{\delta}{\delta A_\nu(y)} \cdot e^{\mathfrak{D}_A} \cdot \frac{e^{L[A]}}{\langle S \rangle} \Big|_{A\to 0} ,$$

where :

$$\mathfrak{D}_A = -\frac{i}{2} \iint \frac{\delta}{\delta A_\mu} D_{c,\mu\nu} \frac{\delta}{\delta A_\nu} , \quad e^{\mathfrak{D}_A} = \text{"linkage op."},$$

$$\langle S \rangle = e^{\mathfrak{D}_A} \cdot e^{L[A]} \Big|_{A\to 0} .$$

<u>NB : $L[A] \Rightarrow M[F]$</u> , $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ .

$\therefore \dfrac{\delta L}{\delta A_\mu(x)} = ie \langle j_\mu(x) \rangle_A$ , $\underline{\partial_\mu \langle j_\mu(x) \rangle_A = 0}$

<u>requires</u> : $\partial_\mu \dfrac{\delta L}{\delta A_\mu} = 0$ . $\therefore$ <u>Automatically</u> :

$\partial_\mu K_{\mu\nu} = \partial_\nu K_{\mu\nu} = 0$ ; $\therefore \underline{\tilde{K}_{\mu\nu}(k) = (k_\mu k_\nu - \delta_{\mu\nu} k^2)\, \Pi(k^2)}$ .

<u>We must evaluate</u> : $Z_3^{-1} = 1 + \Pi(0)$ .

<u>NB'</u> : <u>This</u> quantity, $\Pi(0)$, can be identified and extracted from the FIs <u>before</u> the FIs are performed.

<u>Not true</u> for $\Pi(k^2)$ ...

$$K_{\mu\nu} = -i\, e^{\mathfrak{D}_A}\left(\frac{\delta^2 L}{\delta A_\mu \delta A_\nu} + \underbrace{\frac{\delta L}{\delta A_\mu}\cdot\frac{\delta L}{\delta A_\nu}}_{\text{do not contribute to } Z_3^{-1}}\right)\cdot \frac{e^{L}}{\langle S\rangle}\Bigg|_{A\to 0}.$$

∴ We here consider only:

$$e^{\mathfrak{D}_A}\cdot\frac{\delta^2 L}{\delta A_\mu \delta A_\nu}\, e^{L}\Big|_0 \equiv \left(e^{\mathfrak{D}_A}\frac{\delta^2 L}{\delta A_\mu \delta A_\nu}\right) e^{\overleftrightarrow{\mathfrak{D}}_A}\left(e^{\mathfrak{D}_A} e^{L}/\langle S\rangle\right)\Big|_{A\to 0}$$

where: $\overleftrightarrow{\mathfrak{D}}_A = -i\int \frac{\overleftarrow{\delta}}{\delta A} D_c \frac{\overrightarrow{\delta}}{\delta A}$, $e^{\overleftrightarrow{\mathfrak{D}}_A}$ is the "cross-linkage" operator.

- Introduce (exact, modified) Fradkin rep.:

$$L[A] = -\tfrac{1}{2}\int_0^\infty \frac{ds}{s} e^{-ism^2}\int d^4x'\cdot N\cdot\int d[u]\, e^{\frac{i}{2}\int_0^s u(2h)^{-1}u}\cdot \delta(u(s))\cdot$$
$$\cdot\, e^{-ig_0\int_0^s ds'\, u'_\mu(s')A_\mu(x'-u(s'))}\cdot \mathrm{tr}\left(e^{g\int_0^s ds'\sigma_{\mu\nu}F_{\mu\nu}(x'-u(s'))}\right)_+ ,$$

where $s' =$ proper-time$)^2$ (of "looping" electron), $u_\mu(0) = 0 = u_\mu(s)$,

$N^{-1} = \int d[u]\, e^{\frac{i}{2}\int u(2h)^{-1}u}$, $u_\mu(s')$ is config. space variable of dim. $\ell$,

$h(s_1,s_2) = \theta(s_1-s_2)s_2 + \theta(s_2-s_1)s_1 = \frac{1}{2}\left[s_1+s_2-|s_1-s_2|\right]$,

$$\langle s_1|h^{-1}|s_2\rangle = \frac{\overrightarrow{\partial}}{\partial s_1}\delta(s_1-s_2)\frac{\overleftarrow{\partial}}{\partial s_2}.$$

- Special Cancelations: $e^{\mathfrak{D}_A}\left(e^{g\int_0^s \sigma\cdot F}\right)_+\Big|_{A\to 0} = 1$,

plus other terms associated with this property.
A surprise! Proven (to all orders in $g$) in ANPAEQED I.

Results for $Z_3^{-1}$: a) Lowest order ($\alpha_0$) pert. theory:

$$Z_3^{-1} \to 1 + \frac{\alpha_0}{3\pi}\int_\epsilon^\infty \frac{ds}{s} e^{-ism_0^2} \xrightarrow[s=\epsilon x]{} 1 + \frac{\alpha_0}{3\pi}\int_1^\infty \frac{dx}{x} e^{-i\epsilon m_0^2 x}$$

E.G.: A typical "linkage" operation is:

$$e^{\mathcal{D}_A} \cdot e^{-ie_0\int_0^s ds_1\, y'_\mu(s_1) A_\mu(y-u(s_1))} \Big|_{A\to 0}$$

$$= e^{-\frac{i}{2}\int \frac{\delta}{\delta A} D_c \frac{\delta}{\delta A}} \cdot e^{-ie_0\int_0^s ds'\, u'\cdot A(y-u(s'))} \Big|_{A\to 0}$$

$$= e^{+\frac{i}{2}e_0^2 \int_0^s ds_1 \int_0^s ds_2\, y'_\mu(s_1) D_c^{\mu\nu}(u(s_1)-u(s_2))\, u'_\nu(s_2)}$$

All calculations here are completely gauge-inv., so let's use the simplest Feynman gauge $D_{c\,\mu\nu}$:

$$D_c^{\mu\nu}(z) = \delta_{\mu\nu} D_c(z) = \frac{i}{4\pi^2}\delta_{\mu\nu}\frac{1}{z^2+i\epsilon}\Big|_{\epsilon\to 0}.$$

[In mom. space: $\tilde{D}_c(k) = \frac{1}{k^2-i\delta}\Big|_{\delta\to 0}$.]

In config. space, $\dim.\epsilon = \ell^2$; and we can use $\epsilon$ as a cut-off parameter:

$$\epsilon\big|_{c.s.} \leftrightarrow \Lambda^{-2}\big|_{m.s.}$$

→ As long as $\epsilon \neq 0$, all rad. corrections are finite.

∴ Consider:
$$-\frac{e_0^2}{8\pi^2}\iint_0^s ds_1\, ds_2 \frac{u'(s_1)\cdot u'(s_2)}{(u(s_1)-u(s_2))^2+i\epsilon},$$

and ask:

from where do the UV divs. arise?

$$-\left(\frac{\alpha_0}{2\pi}\right)\iint_0^s ds_1\, ds_2 \frac{u'\cdot u'}{(\Delta u)^2+i\epsilon} \xrightarrow[\text{by symm.}]{} -\left(\frac{\alpha_0}{\pi}\right)\int_0^s ds_1 \int_0^{s_1} ds_2 \frac{u'\cdot u'}{(\Delta u)^2+i\epsilon}.$$

Intuitively : Expect divs. to appear when $u(s_1) \leftrightarrow u(s_2)$.

E.G., in Mom. Space : $D_c(z) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik\cdot z}}{k^2 - i\delta} \Big|_{\delta \to 0}$ ,

and, here, $z = \Delta u = u(s_1) - u(s_2)$.

Combine this $D_c(z)$ with another prop. (e.g., $S_c(z)$) , to form :  . After taking Fourier transform, one finds :

$$\sim \int \frac{d^4k}{k^2 - i\delta} \frac{e^{ik\cdot \Delta u}}{(k-p)^2 + m^2 - i\delta} .$$

When will one see a log. div. ? When $u(s_1) \leftrightarrow u(s_2)$ !

→ This example :  shows exact equivalence of our DP Method with the conventional F. graph calc.; the log. div. is exactly the same.

→ To obtain the log. div. :

In the FI $\int d[u]$ , higher derivatives of $u(s')$ can have wild fluctuations, but $u(s')$ and $u'(s')$ are cont. Also, $\infty$ higher derivatives do not contribute to our integral. ∴ Treat the $u(s')$ as cont. $f(s')$ :

$$\frac{u'(s_1) \cdot u'(s_2)}{(u(s_1) - u(s_2))^2 + i\epsilon} \rightarrow \frac{u'^2(s_1)}{u'^2(s_1)\,(s_1 - s_2)^2 + i\epsilon} \rightarrow \frac{1}{(s_1 - s_2)^2 + i\epsilon \cdot \bar{\epsilon}(s_1)} ,$$

where $\bar{\epsilon}(s_1)$ = sign of $u'^2(s_1) = u'_\mu(s_1) \cdot u'_\mu(s_1)$.

$\therefore$ <u>We require</u> : $-\left(\frac{\alpha_0}{\pi}\right)\int_0^s ds_1 \int_0^{s_1} ds_2 \frac{1}{(s_1-s_2)^2 + i\epsilon \cdot \bar{\epsilon}(s_1)}$

$$\rightarrow -\left(\frac{\alpha_0}{\pi}\right)\int_0^s ds_1 \int_0^{s_1} ds_2 \frac{1}{(s_1 - s_2 + i\epsilon \cdot \bar{\epsilon}(s_1))^2}$$

$$= -\left(\frac{\alpha_0}{\pi}\right)\int_0^s ds_1 \left[ \frac{1}{i\epsilon \cdot \bar{\epsilon}(s_1)} - \frac{1}{s_1 + i\epsilon \cdot \bar{\epsilon}(s_1)} \right]$$

<u>1st term :</u> $\sim \frac{1}{\epsilon} \cdot \int_0^s ds_1\, u'^2(s_1) \rightarrow 0$, for 3 reasons :

(i) $\int d[u]$ makes no ref. to sign of $u'^2$; $\therefore$ as much + as −, & $\therefore$ 0.

(ii) Calc. $\langle \bar{\epsilon}\left(\int_0^s ds_1\, u'^2(s_1)\right) \rangle \rightarrow \int d[u] \cdots \bar{\epsilon}\left(\int_0^s ds_1\, u'^2\right) \Rightarrow 0$, which means : as much + as −, & $\therefore$ 0.

(iii) $\frac{1}{\epsilon} \rightarrow \Lambda^2$, and there are NO such quad. divs. in any order of pert. theory.

$\therefore$ <u>1st term $\Rightarrow$ 0,</u> and :

$$\frac{i}{2} e_0^2 \iint u' \cdot D_c \cdot u' \rightarrow + \left(\frac{\alpha_0}{\pi}\right)\int_0^s ds_1 \frac{1}{s_1 + i\epsilon \cdot \bar{\epsilon}(s_1)}$$

$\hookrightarrow \bar{\epsilon}(0)$, since only $s_1 = 0$ is relevant to <u>this</u> $\int_0^s ds_1$.

$\hookrightarrow \frac{\alpha_0}{\pi} \ln\left(\frac{s}{i\epsilon \cdot \bar{\epsilon}(0)}\right)$.

In pert. approxs., $\bar{\epsilon}(0) \rightarrow -1$; & $\therefore$ we choose <u>consistently</u>, $\bar{\epsilon}(0) = -1$.

$$\therefore\ e^{\frac{i}{2}e_0^2 \iint_0^s u' \cdot D_c \cdot u'}\Big|_{D.P.} = e^{\left(\frac{\alpha_0}{\pi}\right)\left[\ln\left(\frac{s}{\epsilon}\right) + i\frac{\pi}{2}\right]}$$

Set : $p = \frac{\alpha_0}{\pi}$ : $\hookrightarrow e^{i\pi p/2} \cdot \left(\frac{s}{\epsilon}\right)^p$

∘ From under the F.I. $\int d[u] \ldots$ , we've extracted the D.P. of every term arising from the linkages

$$e^{\mathfrak{D}_A} \cdot e^{-ie_0 \int_0^s u' \cdot A(y - u(s'))}\Big|_{A\to 0} .$$

Other simplifications :

In this DP Model, the cross-linkages

$$e^{-ie_0 \int u' \cdot A} \cdot e^{\overleftrightarrow{\mathfrak{D}_A}} \cdot \left(e^{g\int \sigma \cdot F}\right)_+ \Big|_{A\to 0}$$

give : 0 effect : 1 .

∴ In "quenched" approx., neglecting $\exp[L[A]]$ ,

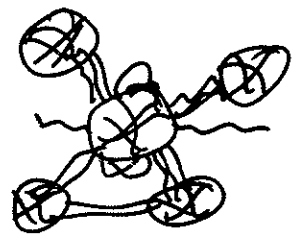
we have extracted all log divs. :

b) For : $\sum$  : $Z_3^{-1} \to 1 + \frac{p}{3}\int_{\epsilon}^{\infty} \frac{ds}{s}\left(\frac{s}{\epsilon}\right)^{p} e^{\frac{i\pi p}{2}}\, e^{-ism_0^2}$ ,

where : $p = \frac{\alpha_0}{\pi}$ , $\alpha_0 = \frac{g_0^2}{4\pi}$ , $\epsilon$ in config. space $\leftrightarrow \frac{1}{\Lambda^2}$ in mom. space

This $Z_3^{-1}$ still divergent ; real but divergent.

c) Adding all closed-electron-loops, interacting with $\frac{\delta^2}{\delta A \delta A}$ :

$\sum$  : Easy to write functionally, impossible to include using FGs.

One simplifying approx. : Contributions of connected loops appear to be very small, and are neglected ; in terms of the functional cluster expansion :

$$e^{\mathfrak{D}_A} e^{L[A]} \equiv e^{\sum_{n=1}^{\infty} Q_n/n!} , \quad Q_n = e^{\mathfrak{D}_A}[L[A]]^n\Big|_{\text{connected},\, A\to 0}.$$

E.G. :

$$Q_1 = e^{\mathfrak{D}_A} L\,\Big|_0 \equiv \bar{L} , \quad Q_2 = \bar{L}[A]\left(e^{\overleftrightarrow{\mathfrak{D}}_A} - 1\right)\bar{L}[A]\Big|_0 , \ \text{etc.},$$

and we use only $Q_1$ .

One qualification : Introduce parameter $\xi \sim O(1)$ , needed when extracting non-quenched closed-loop contributions – a measure of lack of precision. But this introduces one new parameter into the problem :

Instead of : $Z_3^{-1}(\alpha_0) \to Z_3^{-1}(\alpha_0, \xi)$ ,

and $\therefore$ permit a "spread" of $\alpha_0$ values – see this in subsequent pages

How to include all the  closed fermion loops?

By F.Gs, impossible!

Functionally: Obvious, since: $e^{-ie_0\int u'\cdot A}\, e^{\overleftrightarrow{\mathfrak{D}}_A}\, e^{\bar{L}[A]}\Big|_{A\to 0}$

$$= e^{-ie_0(-i)\int u'\cdot D_c\cdot\frac{\delta}{\delta A}}\cdot e^{\bar{L}[A]}$$

$$\Rightarrow e^{\bar{L}[A-e_0\int u'\cdot D_c]},$$

and each $\bar{L}[A]\sim N\int d[v]\ldots\int d^4x''\cdot e^{-ie_0\int_0^t dt'\, v_\nu'(t')\cdot A_\nu(x''-v(t'))}$,

which generates under each $\int d[v]\ldots$ the quantity

$$e^{+ie_0^2\int_0^s ds'\int_0^t dt'\, u_\mu'(s')D_c^{\mu\nu}(x''-y+u(s')-v(t'))\,v_\nu'(t')};$$

and it is this s-dependence, under the FI of each $\bar{L}[A]$ which provides a convergence factor for the original $\int_\epsilon^\infty ds$.

One difficulty: For large $|x''-y|\gg|u-v|$, integral $\to 0$, since $\int_0^s ds'\, u_\mu'(s')=0=\int_0^t dt'\, v_\nu'(t')$.

[the statements: $u(s)=u(0)=0$, $v(t)=v(0)=0$, are the expressions of how MGI is realized.]

But for small $|x-y|$, $|x-y|\ll|u-v|$, we get previous forms, for which DP extraction is possible.

<u>Our procedure</u> : Introduce idealization :

Assume only $\int d^4(x''-y)$ non-zero contributions appear in 4-volume "roughly" of radius $\sqrt{\epsilon}\cdot\xi$ :

$$\int d^4(x''-y) \to 2\pi^2 \cdot \epsilon^2 \cdot \xi^4 \,, \quad \text{where } \xi \lesssim 1 \,.$$

Then, use the DP procedure to extract and sum log divs., and finally, calculate :

$$\int d[v] \cdots \sim \epsilon^2 \int_\epsilon^\infty \frac{dt}{t^3} e^{-im_0^2 t} \cdot \left(\frac{s}{\epsilon}\right)^p \left(\frac{t}{\epsilon}\right)^p \cdot \left(\frac{t}{\epsilon}\right)^p$$

<u>Now set</u> : $x = s/\epsilon$ , $y = t/\epsilon$ , and obtain :

$$Z_3^{-1} = 1 + \frac{p}{3} e^{i\pi p/2} \int_1^\infty dx \cdot x^{p-1} e^{-ix\epsilon m_0^2} \cdot e^{T(x)} \,,$$

$$T(x) = i\left(\frac{\xi}{2}\right)^4 e^{3i\pi p/2} \cdot x^p \int_1^\infty \frac{dy}{y^3} y^{2p} \cdot e^{-iy\epsilon m_0^2} \,.$$

<u>Without $T(x)$</u> : $Z_3^{-1}$ diverges as $\epsilon \to 0$. <u>Now</u> : $\epsilon \to 0$

With $T(x)$ : To have : $\int^\infty dy < \infty$ , $p < 1$ ;

and, by definition, $p > 0$.

$\therefore$ For $0 < p < 1$ we can now let $\epsilon \to 0$ and and obtain a finite $Z_3^{-1}$.

**NB**: as $\epsilon \to 0$, $\epsilon \cdot m_0^2 \to 0$ (true in every order of pert. theory, since $m_0 \sim$ log. div.), and our conclusions (below) hold indep. of mass of charged fermion: any ch. fermion satisfying QED must have the same renormalized charge.

In fact, the $\int^{\infty} dx$ is a perfect differential, and yields:

$$Z_3^{-1} = 1 + \frac{2}{3}(1-p)\left(\frac{\xi}{2}\right)^{-4} i\, e^{-i\pi p} \cdot e^{-g},$$

where:

$$g = -i\left(\frac{\xi}{2}\right)^4 \cdot \frac{e^{3i\pi p/2}}{2(1-p)}.$$

Since we expect $\xi \lesssim 1$, $\left(\frac{\xi}{2}\right)^4 \ll 1$, $e^{-g} \simeq 1 + \dots$, so that:

$$Z_3^{-1} \simeq 1 + \frac{2}{3}(1-p)\left(\frac{\xi}{2}\right)^{-4}\big(\sin(\pi p) + i\cos(\pi p)\big).$$

$\therefore$ for **real** $Z_3$, $p \to \frac{1}{2}$;

and if: $\alpha \equiv \alpha_0 Z_3 = \pi p Z_3 = 1/137$,

then: $\xi \simeq 0.397$, $\left(\frac{\xi}{2}\right)^4 \simeq 0.00155$.

$\therefore$ Any corrections to these $p = \frac{1}{2}$, $\xi \sim \frac{2}{5}$ will be a few parts per thousand.

QED!

- Have we <u>proven</u> that $\alpha = 1/137$ ?

<u>NO</u> — But if it can be done — a proper determination of $\xi$ — then this is the proper approach.

- Does QED have a finite, $g$-inv. sector? Yes! And with $\alpha_0$ finite and $> \alpha$.

- Could $Z_2$ and $\delta m$ also be finite? Perhaps...
- Is QCD Color-charge renormalization finite?

[Relevant functional operations are similar...]

<u>PREDICTION</u> : QFTh is entering a new phase, from its <u>Adolescence</u> of Feynman Graphs in Pert. theory, to a new <u>Maturity</u> based on the use of Schwinger functional sol'ns plus Fradkin representations.