On the renormalizing series of some integral equations

B. Candelpergher and T. Grandou

Toy models

[1] D. Kreimer, *Knots and Feynman Diagrams* (Cambridge University Press, 2000).

[2] A. Connes and D. Kreimer, *Comm. Math. Phys.* **199**, 203 (1998).

EXAMPLE OF INTEGRAL EQUATIONS

EXAMPLE 1 : Integral equation where $x > 0$ (mass parameter).

$$f(x,h) = 1 + h \int_0^{+\infty} \frac{f(y,h)}{y+x}\, dy$$

Solve in the form of a perturbative series

$$f(x,h) = 1 + \sum_{n \geq 1} h^n f_n(x)$$

the $f_n(x)$ are divergent integrals

$$f_n(x) = \int_0^{+\infty} .. \int_0^{+\infty} \frac{1}{y_n + y_{n-1}} \frac{1}{y_{n-1} + y_{n-2}} \; .. \; \frac{1}{y_1 + x}\, dy_1 .. \, dy_n$$

REGULARIZATION

$$f(x,t,h) = 1 + h \int_0^{+\infty} f(y,t,h) \frac{y^{-t}}{y+x}\, dy \;, \qquad \text{where} \quad Re(t) > 0$$

$$f_0(x,t) = 1$$

$$f_1(x,t) = \int_0^{+\infty} \frac{y^{-t}}{y+x}\, dy = x^{-t} \int_0^{+\infty} \frac{u^{-t}}{u+1}\, du = x^{-t} b(t),$$

WHERE

$$b(t) = \frac{\pi}{\sin(\pi t)},$$

BY INDUCTION,

$$f_n(x,t) = A_n(t) x^{-nt} \ , \qquad A_n(t) = b(t)b(2t)...b(nt)$$

AT $t \to 0$, THE f_n DISPLAY MULTIPOLE.

RENORMALIZATION: MINIMAL SUBTRACTION SCHEME

$$f^R(x,t,h) = 1 + h \int_0^{+\infty} f^R(y,t,h) \frac{y^{-t}}{y+x}\, dy - Z(x,t,h)$$

$$Z(x,t,h) = \sum_{n \geq 1} h^n Z_n(x,t).$$

$$f^R(x,t,h) = 1 + \sum_{n \geq 1} h^n f_n^R(x,t)$$

$$f_n^R(x,t) = \int_0^{+\infty} f_{n-1}^R(y,t) \frac{y^{-t}}{y+x}\, dy - Z_n(x,t)$$

FOR $n = 1$

$$f_1^R(x,t) = -Z_1(x,t) + b(t)x^{-t}$$

$$Z_1(x,t) = \text{sing}(b(t)x^{-t}) = \frac{1}{t}$$

SING$(g(t))$ =POLAR PART OF A MEROMORPHIC FUNCTION g AT $t = 0$,

$$f_2^R(x,t) = -Z_2(x,t) + b(t)b(2t)x^{-2t} - \frac{1}{t}b(t)x^{-t}$$

$$Z_2(x,t) = \text{sing}\left(b(t)b(2t)x^{-2t} - \frac{1}{t}b(t)x^{-t}\right) = \frac{-1}{2t^2}$$

THIS DEFINES A RECURSIVE WAY OF CALCULATING THE Z_n, GETTING FOR EXAMPLE

$$Z_3(x,t) = \frac{1}{6t^3} + \frac{1}{18}\frac{\pi^2}{t}$$

$$Z_4(x,t) = -\frac{1}{24t^4} - \frac{1}{18}\frac{\pi^2}{t^2}$$

$$Z_5(x,t) = \frac{1}{120t^5} + \frac{}{36}\frac{\pi^2}{t^3} + \frac{3}{200}\frac{\pi^4}{t}$$

THE $Z_n(x,t)$ WOULD BE x-INDEPENDENT. THE RENORMALIZATION CONSTANTS DO NOT DEPEND ON MASS PARAMETER.

EXAMPLE 2 :

$$f(x,t) = h \int_0^{+\infty} \frac{e^{-x/y}}{y} y^{-t} \; f(y,t) \, dy + 1$$

$$b(t) = \Gamma(t)$$

$$f_n(x,t) = A_n(t) x^{-nt}, \qquad A_n(t) = \Gamma(t)\Gamma(2t)...\Gamma(nt)$$

$$Z_1(x,t) = \frac{1}{t}$$

$$Z_2(x,t) = -\frac{1}{2t^2} - \frac{1}{2t}\gamma$$

$$Z_3(x,t) = \frac{1}{6t^3} + \frac{\gamma}{4}\frac{1}{t^3} + (\frac{\zeta(2)}{6} + \frac{\gamma^2}{2})\frac{1}{t}$$

$$Z_4(x,t) = -\frac{1}{24t^4} - \frac{\gamma}{4}\frac{1}{t^3} - (\frac{5\gamma^2}{8} + \frac{\zeta(2)}{6})\frac{1}{t^2} - (\frac{2\gamma^3}{3} + \frac{\gamma\zeta(2)}{2} + \frac{\zeta(3)}{12})\frac{1}{t}$$

WHERE γ IS THE EULER CONSTANT.

GENERAL CASE : The integral equation

$$f(x,h) = 1 + h\int_0^{+\infty} f(y,h)\frac{1}{x}k(\frac{y}{x})\ dy$$

where $x > 0$.

For $0 < Re(t) < 1$, regularized integral equation

$$f(x,t,h) = 1 + h\int_0^{+\infty} f(y,t,h)y^{-t}\ \frac{1}{x}k(\frac{y}{x})\ dy\ ,$$

define

$$b(t) = \int_0^{\infty} k(u)u^{-t}\ du$$

and assume that the function b has an analytic continuation in a neighbourhood of $t = 0$, with a simple pole at $t = 0$:

$$b(t) = \frac{b_{-1}}{t} + \sum_{p\geq 0} b_p t^p$$

Theorem 1 In the Minimal Subtraction scheme (MS), we have the renormalizing series

$$Z(t,h) = \sum_{n\geq 1} h^n Z_n(t)$$

the $Z_n(t)$ are polynomials of degre n in $1/t$ that do not depend on x.

$$Z_n(t) = \mathrm{sing}\left(A_n(t) - \sum_{k=1}^{n-1} Z_{n-k}(t) A_k(t) \right)$$

where

$$A_n(t) = b(t)b(2t)\ ..\ b(nt)$$

writing

$$Z_n(t) = \sum_{k=1}^{n} \frac{Z_n^{(k)}}{t^k}$$

analyze the convergence of the series

$$\sum_{n\geq 1} h^n Z_n^{(k)}$$

On the first example at $k=1$, the series $\sum_{n\geq 1} h^n Z_n^{(1)}$ has a convergence radius of $R^{(1)} \simeq 1/\sqrt{9.86960439} \simeq 1/3.1416$.

ALGEBRAIC MODEL

$Z_n(t)$ IS COMPLETLY DETERMINED BY $A_n(t)$ AND BY THE CONDITION:

ANALITYCITY AT $t = 0$ OF

$$f_n^R(t) = A_n(t) - \sum_{k=1}^{n-1} Z_{n-k}(t) A_k(t) - Z_n(t)$$

EQUIVALENT TO ANALITYCITY AT $t = 0$ OF

$$(1 - \sum_{n\geq 1} h^n Z_n(t))(1 + \sum_{n\geq 1} h^n A_n(t)) = f^R(t,h) = 1 + \sum_{n\geq 1} h^n f_n^R(t)$$

EQUIVALENT TO THE RENORMALIZATION OF ALGEBRAIC EQUATION

$$f^R(t,h) = c(t,h) f^R(t,h) + 1 - Z(t,h)$$

WHERE

$$c(t,h) = 1 - \frac{1}{1 + \sum_{n\geq 1} h^n A_n(t)} = 1 - \frac{1}{1 + \sum_{n\geq 1} h^n b(t) b(2t)\ ..\ b(nt)}$$

$$c(t,h) \to 1 - \frac{1}{1+\sum_{n\geq 1} h^n (b(t))^n} = hb(t)$$

RENORMALIZATION OF THE EQUATION

$$f^R(t,h) = hb(t)f^R(t,h) + 1 - Z(t,h)$$

$Z_n(t)$ WITH A SIMPLE POLE AT THE ORIGIN

$$Z_n(t) = \frac{Z_n^1}{t}$$

$$\frac{Z_n^1}{t} = \text{sing}\left(b^n(t) - \sum_{k=1}^{n-1} b^k(t)\frac{Z_{n-k}^1}{t}\right)$$

Theorem 2 THE RENORMALIZING SERIES DEFINES A FUNCTION

$$Z^1(h) = \sum_{n\geq 1} Z_n^1 h^n$$

WHICH, IN A NEIGHBOURHOOD OF $h = 0$, IS THE INVERSE OF THE FUNCTION $t \mapsto 1/b(t)$

$$Z^1(h) = t \leftrightarrow h = 1/b(t)$$

THE CLASSICAL LAGRANGE INVERSION FORMULA GIVES

$$Z_n^1 = \frac{1}{n!}\partial_t^{n-1}((tb(t))^n(t))_{|t=0} = \frac{1}{n!}\partial_t^{n-1}\left((b_{-1} + \sum_{p\geq 0} b_p t^{p+1})^n(t)\right)_{|t=0}$$

A SURPRISING RELATION

EXAMPLE 1:

$$b(t) = \frac{\pi}{\sin(\pi t)}$$

$Z^1(h)$ IS THE INVERSE OF THE FUNCTION $t \mapsto \frac{\sin(\pi t)}{\pi}$.

$$Z^1(h) = \frac{1}{\pi}\arcsin(\pi h) = \int_0^h \frac{\mathrm{d}s}{\sqrt{(1-\pi^2 s^2)}}$$

THE CONVERGENCE RADIUS OF THE SERIES CORRESPONDING TO $Z^1(h)$ IS $1/\pi$.

THE FIRST FEW VALUES OF THE Z_n^1

$$Z^1(h) = h + \left(\frac{1}{6}\pi^2\right)h^3 + \left(\frac{3}{40}\pi^4\right)h^5 + \left(\frac{5}{112}\pi^6\right)h^7 + \left(\frac{35}{1152}\pi^8\right)h^9 + ...$$

THE EXPANSION FOR THE PARTIAL $Z^{(1)}$, OF THE FIRST INTEGRAL EQUATION

$$Z^{(1)}(h) = h + \left(\frac{1}{6}\pi^2\right)\frac{h^3}{3} + \left(\frac{3}{40}\pi^4\right)\frac{h^5}{5} + \left(\frac{5}{112}\pi^6\right)\frac{h^7}{7} + \left(\frac{35}{1152}\pi^8\right)\frac{h^9}{9} + .$$

IT APPEARS READILY THAT ONE WOULD HAVE

$$Z_n^{(1)} = \frac{1}{n}Z_n^1$$

AND THUS, FOR THIS EXAMPLE

$$\partial Z^{(1)}(h) = \frac{Z^1(h)}{h} \rightarrow Z^{(1)}(h) = \int_0^h \frac{Z^1(u)}{u}\mathrm{d}u$$

$$Z^{(1)}(h) = \int_0^h \frac{\arcsin \pi u}{\pi u}\mathrm{d}u$$

CONFIRMING THE RADIUS OF $1/\pi$

EXAMPLE 2 :

IT GENERATES THE FUNCTIONS

$$A_n(t) = \Gamma(t)\Gamma(2t)...\Gamma(nt)$$

IN THIS CASE

$$b(t) = \Gamma(t) = \frac{b_{-1}}{t} + b_0 + b_1 t + ..$$

TO THE SIMPLIFIED ALGEBRAIC MODEL

$$f^R(t,h) = h\Gamma(t) f^R(t,h) + 1 - Z(t,h)$$

$Z^1(h)$ IS THE INVERSE OF THE FUNCTION

$$t \mapsto \frac{1}{b(t)} = \frac{1}{\Gamma(t)}$$

BY LAGRANGE FORMULA

$$Z_1^1 = 1, Z_2^1 = -\gamma, Z_3^1 = \frac{3\gamma^2}{2}, Z_4^1 = -2\gamma\zeta(2) + \frac{\zeta(3)}{3}$$

IN THE INTEGRAL MODEL

$$Z_1^{(1)} = 1, Z_2^{(1)} = -\frac{1}{2}\gamma, Z_3^{(1)} = \frac{\gamma^2}{2}, Z_4^{(1)} = -\frac{\gamma\zeta(2)}{2} + \frac{\zeta(3)}{12}$$

THE PROPERTY

$$Z_n^{(1)} = \frac{1}{n} Z_n^1$$

IS VALID FOR ALL OF THE INTEGRAL EQUATIONS OF THE FAMILY DEFINED.

THUS IN THE MS SCHEME

$$Z^{(1)}(h) = \int_0^h \frac{Z^1(u)}{u} \mathrm{d}u$$

WHERE $Z^1(h)$ IS THE INVERSE OF THE FUNCTION

$$t \mapsto \frac{1}{\int_0^\infty k(u) u^{-t}\, du}$$